# A new modified Galerkin method for the two-dimensional Navier-Stokes equations

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February 1, 2008

**Abstract**

We present a new type of modified Galerkin method. It is a construction with several (inductively defined) levels, that provides approximate solutions of increasing accuracy with every new level. These solutions are constructed as approximations of the so called induced trajectories (notion on which the definition of a class of approximate inertial manifolds used in the nonlinear and postprocessed Galerkin methods is based).

Key words: *Navier-Stokes equations, induced trajectories, nonlinear and postprocessed Galerkin methods, approximate inertial manifolds*

MSC AMS 2000: 35K55, 65M60.

## 1 Introduction

We consider the Navier-Stokes equations for a planar flow, with periodic boundary conditions. The functional framework and the basic hypothesis are presented in Section 2.

As usual in the Galerkin method, the space of functions is split into a direct sum of two subspaces: one is the finite dimensional space spanned by the eigenfunctions corresponding to a finite set, $\Gamma_m$, of eigenvalues of the linear operator $\mathbf{A} = -\boldsymbol{\Delta}$ and the other is the orthogonal complement of the first. The solution $\mathbf{u}$ of the Navier-Stokes equations will be projected on these spaces: $\mathbf{u} = \mathbf{Pu} + \mathbf{Qu} = \mathbf{p} + \mathbf{q}$, where $\mathbf{P}$ is the projector on the finite dimensional space, and $\mathbf{Q} = \mathbf{I} - \mathbf{P}$ (Section 3).

In Section 4 we improve the estimates for $\mathbf{q}$ proved in [3]. There, the $\left[L^2(\Omega)\right]^2$ norm of this function is found to be less than $K_0 L^{\frac{1}{2}} \delta$, with $\delta = \frac{\lambda}{\Lambda}$, where $\lambda$ is the least eigenvalue of $\mathbf{A}$, $\Lambda$ is the least eigenvalue of $\mathbf{A}$ not belonging to $\Gamma_m$, and $L$ depends increasingly on the number of eigenvalues in $\Gamma_m$. The presence of $L$ is not convenient for our work, since we construct an iterative approximation processus, and if we would use this estimate, at every step a

factor of $L^{1/2}$ would appear in the evaluation of the error. This would lead us to bad estimates of the accuracy of our approximate solutions. We obtain estimates of $\mathbf{q}$ independent of $L$.

Our modified Galerkin method is presented in Section 5. The first level of the method is related to the already classical postprocessed method [4]. This one consists in correcting the Galerkin approximation of the solution, computed at the end of the time integration interval, $\mathbf{p}_0\,(T)$ (that is, the solution of (24) at $T$), by adding to it the function $\mathbf{q}_0\,(T) = \Phi_0(\mathbf{p}_0\,(T))$. $\Phi_0(\cdot)$ is the function whose graph is the approximate inertial manifold (a.i.m.) defined in [3]. The approximation of $\mathbf{u}\,(T)$ is taken as $\mathbf{p}_0\,(T) + \mathbf{q}_0(T)$. Unlike this "postprocessed" Galerkin method, in our method the function $\mathbf{q}_0\,(t) = \Phi_0(\mathbf{p}_0\,(t))$ is computed at every moment $t$. From the numerical point of view this means that it must be computed at every point of the time grid on $[0, T]$. The approximate solution at every $t$ is $\mathbf{u}_0(t) = \mathbf{p}_0\,(t) + \mathbf{q}_0(t)$. The error (in $\left[L^2_{per}\,(\Omega)\right]^2$) of this approximation is of the order of $\delta^{5/4}$. We must remind here the notion of induced trajectory of [15]. There a family of functions is defined, $\{\mathbf{u}_{j,m}\}_{j\geq 0}$, that approximate the exact solution of the Navier-Stokes equations. The first of these is $\mathbf{u}_{0,m}(t) = \mathbf{p}(t) + \mathbf{q}_{0,m}(t)$, where $\mathbf{q}_{0,m}(t) = \Phi_0(\mathbf{p}\,(t))$, and $\mathbf{p}(t)$ is, as above, the $\mathbf{P}$ projection of the exact solution. So, our function $\mathbf{u}_0(t)$ is an approximation of $\mathbf{u}_{0,m}(t)$.

At the second level we look for a new (and better) approximation $\mathbf{p}_1$ of $\mathbf{p}$, by solving an equation closer to the $\mathbf{P}$ projection of the Navier-Stokes equation than the Galerkin equation. That is, in the equation for $\mathbf{p}_1$, in the argument of the nonlinear term, we approximate $\mathbf{p} + \mathbf{q}$ (that appears in the projected N-S equation (13)) with $\mathbf{p}_1 + \mathbf{q}_0$, with $\mathbf{q}_0$ defined above (equation (27) in Section 5). The initial condition for $\mathbf{p}_1$ is $\mathbf{P}\mathbf{u}_0$. The equation (27) is different from those arising in the non-linear Galerkin methods, since here $\mathbf{q}_0$ is already determined at the preceding step, while in the non-linear Galerkin methods [9], [2] in the equation for $\mathbf{p}$, the argument of the nonlinear term is $\mathbf{p} + \Phi_0(\mathbf{p})$. Solving equation (27) of our method is not essentially more difficult than solving the Galerkin equation.

Then we compute $\mathbf{q}_1\,(t) = \widetilde{\Phi}_1(\mathbf{p}_1\,(t)\,, \mathbf{q}_0\,(t))$ where $\widetilde{\Phi}_1$ is given by (28). The definition of $\mathbf{q}_1\,(.)$ is inspired from that of the function $\mathbf{q}_{1,m}\,(.)$ of [15] and $\mathbf{q}_1\,(.)$ is an approximation of this latter function. There is an obvious connection between the definition of $\mathbf{q}_1\,(t)$ and the second a.i.m. from the family defined in [15] and used in the non-linear Galerkin methods (the definition of this a.i.m. uses that of $\mathbf{q}_{1,m}\,(.)$). The new approximate solution of the Navier-Stokes equations we define is $\mathbf{u}_1\,(t) = \mathbf{p}_1\,(t) + \mathbf{q}_1\,(t)$. This is a better approximation of the exact solution that the preceding one, since the error is of the order of $\delta^{7/4}$ (as is shown in Section 6).

We define inductively the next levels of our method. By assuming that we already found $\mathbf{p}_j$, $\mathbf{q}_j$ for $j \leq k + 1$ $(k \geq 0)$, we construct the equation for $\mathbf{p}_{k+2}$, taking $\mathbf{p}_{k+2} + \mathbf{q}_{k+1}$ in the argument of the nonlinear term (equation (30)). The "small eddies" component of the solution will be approximated by $\mathbf{q}_{k+2}\,(t) = \widetilde{\Phi}_{k+2}(\mathbf{p}_{k+2}\,(t)\,, \mathbf{q}_{k+1}\,(t)\,, \mathbf{q}_k\,(t))$, where $\widetilde{\Phi}_{k+2}$ is a function (given by the right hand side of (31)) whose construction was inspired by that of the

function $\mathbf{q}_{k+2,m}(t)$ of [15]. The error of $\mathbf{u}_{k+2}(t) = \mathbf{p}_{k+2}(t) + \mathbf{q}_{k+2}(t)$ is of the order of $\delta^{5/4+(k+2)/2}$. The set $\{\mathbf{u}_{k+2}(t) \ ; t \geq 0\}$ is an approximation of the induced trajectory $\{\mathbf{u}_{k+2,m}(t) \ ; t \geq 0\}$.

In Section 6 we prove the estimates of the error of our approximate solutions. Finally some comments on the advantages and drawbacks of this method are given in Section 7.

## 2 The equations, the functional framework

The plane flow of an incompressible Newtonian fluid is modelled by the Navier-Stokes equations:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \mathbf{u} + \nabla p &= \mathbf{f}, \quad (1) \\ \mathrm{div} \mathbf{u} &= 0, \quad (2) \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0(\cdot), \quad (3) \end{aligned}$$

where $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ is the fluid velocity, $\mathbf{x} \in \Omega \subset \mathbb{R}^2$, $\mathbf{u}(., \mathbf{x}) : [0, \infty) \to \mathbb{R}^2$, $p(., \mathbf{x}) : [0, \infty) \to \mathbb{R}$ is the pressure of the fluid, $\nu$ is the kinematic viscosity, and $\mathbf{f}$ is the volume force. We take here $\Omega = (0, l) \times (0, l)$ and consider the case of periodic boundary conditions.

The kinematic viscosity is measured in centistokes ($= mm^2/s$). For water at 20 Celsius degrees it's value is around 1. In order to have coherent measure units, we consider the velocity measured in mm/s. We do not focus here on the mechanics of fluids aspects of the problem, but we focus on the mathematical construction of the approximate solution. However, we must remark that the method is appropriate for the study of Newtonian fluids not having very small kinematic viscosity.

We assume that $\mathbf{f}$ is independent of time and is an element of $\left[L^2_{per}(\Omega)\right]^2$. As is usual in the study of the Navier-Stokes equations with periodic boundary conditions, we assume that [13], [12]

$$\bar{\mathbf{f}} = \frac{1}{l^2} \int_\Omega \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \mathbf{0}, \quad (4)$$

and that the pressure is a periodic function on $\Omega$. For simplicity we will assume also that the average of the velocity over the periodicity cell is zero.

The velocity $\mathbf{u}$ is thus looked for in the space $\mathcal{H} = \left\{\mathbf{v};\ \mathbf{v} \in \left[L^2_{per}(\Omega)\right]^2,\ \mathrm{div}\, \mathbf{v} = \mathbf{0},\ \bar{\mathbf{u}} = 0\right\}$. The scalar product in $\mathcal{H}$ is $(\mathbf{u}, \mathbf{v}) = \int_\Omega (u_1 v_1 + u_2 v_2)\, dx$, (where $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$). The induced norm is denoted by $|\quad|$.

We also need the space $\mathcal{V} = \left\{\mathbf{u} \in \left[H^1_{per}(\Omega)\right]^2,\ div\, \mathbf{u} = \mathbf{0}, \bar{\mathbf{u}} = 0\right\}$, with the scalar product $((\mathbf{u}, \mathbf{v})) = \sum_{i,j=1}^2 \left(\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}\right)$, and the induced norms, denoted by $\|\ \|$. We denote $\mathbf{A} = -\boldsymbol{\Delta}$ and observe that $\mathbf{A}$ is defined on $D(\mathbf{A}) = \mathcal{H} \cap H^2(\Omega)$.

We shall focus on finding approximations for the function $\mathbf{u}$.

The classical variational formulation of the Navier-Stokes equations [13] leads to the abstract equation

$$\frac{d\mathbf{u}}{dt} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} = \mathbf{f} \quad \text{in } \mathcal{V}', \tag{5}$$

$$\mathbf{u}\,(0) = \mathbf{u}_0, \;\; \mathbf{u}_0 \in \mathcal{H}. \tag{6}$$

The notations $\mathbf{B}(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}$, $\mathbf{B}(\mathbf{u}) = \mathbf{B}(\mathbf{u},\mathbf{u}),\ \mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = (\mathbf{B}(\mathbf{u},\mathbf{v}),\mathbf{w})$ will be used below.

For the bilinear application $\mathbf{B}(\mathbf{u},\mathbf{v})$ the following inequalities

$$|\mathbf{B}\,(\mathbf{u},\mathbf{v})| \leq c_1\,|\mathbf{u}|^{\frac{1}{2}}\,|\boldsymbol{\Delta}\mathbf{u}|^{\frac{1}{2}}\,\|\mathbf{v}\|\,, \qquad (\forall)\;\; \mathbf{u}\in D(\mathbf{A}),\;\; \mathbf{v}\in\mathcal{V}, \tag{7}$$

$$|\mathbf{B}\,(\mathbf{u},\mathbf{v})| \leq c_2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\left[1+\ln\left(\frac{|\boldsymbol{\Delta}\mathbf{u}|^2}{\lambda_1\,\|\mathbf{u}\|^2}\right)\right]^{\frac{1}{2}},\;(\forall)\;\; \mathbf{u}\in D(\mathbf{A}),\;\; \mathbf{v}\in\mathcal{V}. \tag{8}$$

hold [6], [13], [15]. We remind the following properties of the trilinear form $\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})$ (valid for periodic boundary conditions [12]):

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w}) = -\mathbf{b}(\mathbf{u},\mathbf{w},\mathbf{v}), \tag{9}$$

$$\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \tag{10}$$

as well as the following inequalities [12]

$$|\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})| \leq c_3\,|\mathbf{u}|^{\frac{1}{2}}\,\|\mathbf{u}\|^{\frac{1}{2}}\,\|\mathbf{v}\|\,|\mathbf{w}|^{\frac{1}{2}}\,\|\mathbf{w}\|^{\frac{1}{2}}\,,\;(\forall)\,\mathbf{u},\mathbf{v},\mathbf{w}\in\mathcal{V}, \tag{11}$$

$$|\mathbf{b}(\mathbf{u},\mathbf{v},\mathbf{w})| \leq c_4\,|\mathbf{u}|^{\frac{1}{2}}\,\|\mathbf{u}\|^{\frac{1}{2}}\,\|\mathbf{v}\|^{\frac{1}{2}}\,|\Delta\mathbf{v}|^{\frac{1}{2}}\,|\mathbf{w}|\,,\;(\forall)\,\mathbf{u}\in\mathcal{V},\;\mathbf{v}\in\mathbf{D}\,(\mathbf{A})\,,\mathbf{w}\in\mathcal{H}. \tag{12}$$

For the problem (5), (6) we have the classical existence and uniqueness results for the equations Navier-Stokes in $\mathbb{R}^2$, with periodic boundary conditions.

**Theorem 1** [13]. *a) If* $\mathbf{u}_0 \in \mathcal{H}$, $\mathbf{f} \in \mathcal{H}$, *then the problem* (5), (6) *has an unique solution* $\mathbf{u} \in C^0\,([0,T];\mathcal{H}) \cap L^2\,(0,T;\mathcal{V})$. *b) If, in addition to the hypotheses in a),* $\mathbf{u}_0 \in \mathcal{V}$, *then* $\mathbf{u} \in C^0\,([0,T];\mathcal{V}) \cap L^2\,(0,T;D(\mathbf{A}))$. *The solution is, in this latter case, analytic in time on the positive real axis.*

The semi-dynamical system $\{S\,(t)\}_{t\geq 0}$ generated by problem (5) is dissipative [14], [12]. More precisely, there is a $\rho_0 > 0$ such that for every $R > 0$, there is a $t_0(R) > 0$ with the property that for every $\mathbf{u}_0 \in \mathcal{H}$ with $|\mathbf{u}_0| \leq R$, we have $|S\,(t)\,\mathbf{u}_0| \leq \rho_0$ for $t > t_0(R)$ . In addition, there are absorbing balls in $\mathcal{V}$ and $\mathbf{D}\,(\mathbf{A})$ for $\{S\,(t)\}_{t\geq 0}$, i.e. there are $\rho_1 > 0$, $\rho_2 > 0$ and $t_1(R),\ t_2(R)$ with $t_2(R) \geq t_1(R) \geq t_0(R)$ such that for every $R > 0$, $|\mathbf{u}_0| \leq R$ implies $\|S\,(t)\,\mathbf{u}_0\| \leq \rho_1$ for $t > t_1(R)$ and $|\mathbf{A}S\,(t)\,\mathbf{u}_0| \leq \rho_2$ for $t > t_2(R)$.

# 3 The decomposition of the space, the projections of the equations

The eigenvalues of $\mathbf{A}$ are $\lambda_{j_1,j_2} = \frac{4\pi^2}{l^2}\left(j_1^2 + j_2^2\right), \quad (j_1,\ j_2) \in \mathbb{N}^2 \setminus \{(0,0)\}$, and the corresponding eigenfunctions are

$$\begin{aligned}
\mathbf{w}^{s\pm}_{j_1,j_2} &= \frac{\sqrt{2}}{l}\frac{(j_2, \mp j_1)}{|\mathbf{j}|}\sin\left(2\pi\frac{j_1x_1 \pm j_2x_2}{l}\right), \\
\mathbf{w}^{c\pm}_{j_1,j_2} &= \frac{\sqrt{2}}{l}\frac{(j_2, \mp j_1)}{|\mathbf{j}|}\cos\left(2\pi\frac{j_1x_1 \pm j_2x_2}{l}\right),
\end{aligned}$$

where $|\mathbf{j}| = \left(j_1^2 + j_2^2\right)^{\frac{1}{2}}$[15]. These eigenfunctions form a total system for $\mathcal{H}$. In order to be able to write easily sums involving the four eigenfunctions above, we denote them as follows

$$\mathbf{w}^{s+}_{j_1,j_2} = \mathbf{w}^1_{j_1,j_2},\ \ \mathbf{w}^{s-}_{j_1,j_2} = \mathbf{w}^2_{j_1,j_2},\ \ \mathbf{w}^{c+}_{j_1,j_2} = \mathbf{w}^3_{j_1,j_2},\ \ \mathbf{w}^{c-}_{j_1,j_2} = \mathbf{w}^4_{j_1,j_2}.$$

For a fixed $m \in \mathbb{N}$ we consider the set $\Gamma_m$ of eigenvalues $\lambda_{j_1,j_2}$ having $0 \leq j_1, j_2 \leq m$. We define

$$\begin{aligned}
\lambda &: = \lambda_{1,0} = \lambda_{0,1} = \frac{4\pi^2}{l^2}, \\
\Lambda &: = \lambda_{m+1,0} = \lambda_{0,m+1} = \frac{4\pi^2}{l^2}(m+1)^2, \\
\delta &= \delta(m) := \frac{\lambda}{\Lambda} = \frac{1}{(m+1)^2}.
\end{aligned}$$

$\Lambda$ is the least eigenvalue not belonging to $\Gamma_m$. The eigenfunctions corresponding to the eigenvalues of $\Gamma_m$ span a finite-dimensional subspace of $\mathcal{H}$. We denote by $\mathbf{P}$ the orthogonal projection operator on this subspace and by $\mathbf{Q}$ the orthogonal projection operator on the complementary subspace. We write for the solution $\mathbf{u}$ of (5),

$$\mathbf{p} = \mathbf{P}\mathbf{u},\ \mathbf{q} = \mathbf{Q}\mathbf{u}.$$

By projecting equation (5) on the above constructed spaces, we obtain

$$\frac{d\mathbf{p}}{dt} - \nu\Delta\mathbf{p} + \mathbf{P}\mathbf{B}(\mathbf{p}+\mathbf{q}) = \mathbf{P}\mathbf{f}, \tag{13}$$

$$\frac{d\mathbf{q}}{dt} - \nu\Delta\mathbf{q} + \mathbf{Q}\mathbf{B}(\mathbf{p}+\mathbf{q}) = \mathbf{Q}\mathbf{f}. \tag{14}$$

# 4 New estimates for the "small" component of the solution

In [3] is proved that for every $R > 0$, there is a moment $t_3(R) \geq t_2(R)$ such that for every $|\mathbf{u}_0| \leq R$,

$$\begin{aligned} |\mathbf{q}(t)| &\leq K_0 L^{\frac{1}{2}} \delta, \quad \|\mathbf{q}(t)\| \leq K_1 L^{\frac{1}{2}} \delta^{\frac{1}{2}}, \\ |\mathbf{q}'(t)| &\leq K_0' L^{\frac{1}{2}} \delta, \quad |\Delta \mathbf{q}(t)| \leq K_2 L^{\frac{1}{2}}, \qquad t \geq t_3(R), \end{aligned} \tag{15}$$

where $K_0,\ K_0',\ K_1,\ K_2$ depend of $\nu,\ |f|,\ \lambda$ and, for the way we chose the projection subspaces, $L = L(m) = 1 + \ln 2m^2$ (see also [15]). The constant $L$ comes from the use of inequality (8) in the course of the proof of (15). More specific

$$\begin{aligned} L &= \sup_{\mathbf{p} \in \mathbf{P}\mathcal{H}} \left( 1 + \ln \frac{|\Delta \mathbf{p}|^2}{\lambda_1 \|\mathbf{p}\|^2} \right) = \max_{\lambda \in \Gamma_m} \left( 1 + \ln \frac{\lambda}{\lambda_1} \right) = \\ &= 1 + \ln 2m^2. \end{aligned}$$

In the sequel we shall improve the above estimates, trying to eliminate $L$ (which tends to infinity with $m$) from the constants. The idea is that of refining the contribution of the term $\mathbf{QB}(\mathbf{p})$ resulting from $\mathbf{QB}(\mathbf{p}+\mathbf{q})$ in (14). We start from the trigonometric relation

$$\begin{aligned} \sin\left(2\pi \frac{j_1 x_1 \pm j_2 x_2}{l}\right) \sin\left(2\pi \frac{k_1 x_1 \pm k_2 x_2}{l}\right) &= \frac{1}{2} \left[ \cos 2\pi \frac{(j_1 - k_1) x_1 \pm (j_2 - k_2) x_2}{l} - \right. \\ &\quad \left. - \cos 2\pi \frac{(j_1 + k_1) x_1 \pm (j_2 + k_2) x_2}{l} \right], \end{aligned}$$

and the similar ones for all other combinations of sine and cosine that might appear in the scalar product of two eigenfunctions. Since $\mathbf{p} = \sum\limits_{0 \leq j_1, j_2 \leq m} \sum\limits_{i=1}^{4} p_{j_1,j_2}^i \mathbf{w}_{j_1,j_2}^i$, from $(\mathbf{p}\boldsymbol{\nabla})\,\mathbf{p} = \left( \sum\limits_{0 \leq j_1, j_2 \leq m} \sum\limits_{i=1}^{4} p_{j_1,j_2}^i \mathbf{w}_{j_1,j_2}^i \nabla \right) \left( \sum\limits_{0 \leq k_1, k_2 \leq m} \sum\limits_{l=1}^{4} p_{k_1,k_2}^l \mathbf{w}_{k_1,k_2}^l \right)$, only those products of terms that have $j_1 + k_1 \geq m+1$ or $j_2 + k_2 \geq m+1$ will belong to $\mathbf{Q}\mathcal{H}$.

We consider from this point on that $m$ is even, and we set $m = 2n$.

If for $\mathbf{w}_{j_1,j_2}^i$ and $\mathbf{w}_{k_1,k_2}^l$ we have $j_1, j_2 \leq n$ and $k_1, k_2 \leq n$, then $\left(\mathbf{w}_{j_1,j_2}^i \nabla\right) \mathbf{w}_{k_1,k_2}^l$ belongs to $\mathbf{P}\mathcal{H}$. We are led to the idea of considering the subspace of $\mathcal{H}$ spanned by all the eigenfunctions $\mathbf{w}_{j_1,j_2}^i$ with $0 \leq j_1,\ j_2 \leq n$, $1 \leq i \leq 4$. We denote by $\mathbf{P_p}$ the projection operator on this space and set $\mathbf{P_q} = \mathbf{P} - \mathbf{P_p}$, $\mathbf{p}_p = \mathbf{P_p p}$ and $\mathbf{p}_q = \mathbf{P_q p}$. Obviously

$$\mathbf{Q}\left(\mathbf{p}_p \boldsymbol{\nabla}\right) \mathbf{p}_p = \mathbf{0}.$$

On another hand, we see that $\mathbf{p}_q$ is a truncation of $(\mathbf{I}-\mathbf{P_p})\mathbf{u}$, hence $|\mathbf{p}_q| \le |(\mathbf{I}-\mathbf{P_p})\mathbf{u}|$ , $\|\mathbf{p}_q\| \le \|(\mathbf{I}-\mathbf{P_p})\mathbf{u}\|$ . Then, by setting $\delta_1 = \delta(n) = \frac{1}{(n+1)^2}$, $L_1 = L(n) = 1 + \ln 2n^2$, the estimates (15) imply

$$
\begin{aligned}
|\mathbf{p}_q| &\le K_0 L_1^{\frac{1}{2}} \delta_1, \quad \|\mathbf{p}_q\| \le K_1 L_1^{\frac{1}{2}} \delta_1^{\frac{1}{2}}, && (16)\\
\left|\mathbf{p}_q'\right| &\le K_0' L_1^{\frac{1}{2}} \delta_1, \quad |\Delta \mathbf{p}_q| \le K_2 L_1^{\frac{1}{2}}. && (17)
\end{aligned}
$$

We use these inequalities in order to refine the estimates (15). In the rest of the paper we shall assume that for a fixed $R \ge 0$, the function $\mathbf{u}_0$ is such that $|\mathbf{u}_0| \le R$. We state and prove

**Theorem 1.** *There are some constants $\widetilde{C}_0$, $\widetilde{C}_1$, $\widetilde{C}'_0$, $\widetilde{C}_2$, depending only on $\nu$, $\lambda$, $|\mathbf{Qf}|$ such that, for $t$ large enough, the inequalities*

$$
\begin{aligned}
|\mathbf{q(t)}| &\le \widetilde{C}_0 \delta, && (18)\\
\|\mathbf{q(t)}\| &\le \widetilde{C}_1 \delta^{\frac{1}{2}}, && (19)\\
|\mathbf{q}'(\mathbf{t})| &\le \widetilde{C}'_0 \delta, && (20)\\
|\Delta \mathbf{q(t)}| &\le \widetilde{C}_2, && (21)
\end{aligned}
$$

*hold.*

**Proof.** We have, with the notation settled before the Proposition,

$$
\begin{aligned}
\mathbf{QB(p)} &= \mathbf{QB}(\mathbf{p}_p + \mathbf{p}_q) = \mathbf{QB}(\mathbf{p}_p) + \mathbf{QB}(\mathbf{p}_p, \mathbf{p}_q) + \mathbf{QB}(\mathbf{p}_q, \mathbf{p}_p) + \mathbf{QB}(\mathbf{p}_q)\\
&= \mathbf{QB}(\mathbf{p}_p, \mathbf{p}_q) + \mathbf{QB}(\mathbf{p}_q, \mathbf{p}_p) + \mathbf{QB}(\mathbf{p}_q), && (22)
\end{aligned}
$$

since $\mathbf{QB}(\mathbf{p}_p) = 0$. Now, as is usual, we take the scalar product of (14) with $\mathbf{q}$, and by using (10) and (22), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d\,|\mathbf{q}|^2}{dt} + \nu \|\mathbf{q}\|^2 &\le \left|\left(\mathbf{B}(\mathbf{p}_p, \mathbf{p}_q), \mathbf{q}\right)\right| + \left|\left(\mathbf{B}(\mathbf{p}_q, \mathbf{p}_p), \mathbf{q}\right)\right| + \\
&\quad + |(\mathbf{B}(\mathbf{p}_q), \mathbf{q})| + |(\mathbf{B}(\mathbf{q}, \mathbf{p}), \mathbf{q})| + |(\mathbf{Qf}, \mathbf{q})| . && (23)
\end{aligned}
$$

For the first term of the right-hand side, the following estimates (obtained by using (7) and (16)) hold

$$
\begin{aligned}
\left|\left(\mathbf{B}(\mathbf{p}_p, \mathbf{p}_q), \mathbf{q}\right)\right| &\le c_1 |\mathbf{p}_p|^{1/2} |\Delta \mathbf{p}_p|^{1/2} \|\mathbf{p}_q\| \, |\mathbf{q}| \le c_1 \rho_0^{1/2} \rho_2^{1/2} K_1 L_1^{\frac{1}{2}} \delta_1^{\frac{1}{2}} \frac{1}{\Lambda^{\frac{1}{2}}} \ \|\mathbf{q}\|\\
&\le c_1^2 \rho_0 \rho_2 K_1^2 L_1 \delta_1 \frac{2}{\nu \Lambda} + \frac{\nu}{8} \|\mathbf{q}\|^2 .
\end{aligned}
$$

For the second term we obtain the inequalities

$$
\begin{aligned}
\left|\left(\mathbf{B}(\mathbf{p}_q, \mathbf{p}_p), \mathbf{q}\right)\right| &\le c_1 |\mathbf{p}_q|^{1/2} |\Delta \mathbf{p}_q|^{1/2} \|\mathbf{p}_p\| \, |\mathbf{q}| \le c_1 K_1^{1/2} L_1^{\frac{1}{4}} \delta_1^{\frac{1}{2}} \rho_2^{1/2} \rho_1 \frac{1}{\Lambda^{\frac{1}{2}}} \ \|\mathbf{q}\|\\
&\le c_1^2 K_1 \rho_1^2 \rho_2 L_1^{1/2} \delta_1 \ \frac{2}{\nu \Lambda} + \frac{\nu}{8} \|\mathbf{q}\|^2 .
\end{aligned}
$$

For the third term we have

$$\begin{aligned}
|(\mathbf{B}(\mathbf{p}_q),\mathbf{q})| &\leq c_2 L^{\frac{1}{2}} \|\mathbf{p}_q\|^2 |\mathbf{q}| \leq c_2 L^{\frac{1}{2}} K_1^2 L_1 \delta_1 \frac{1}{\Lambda^{\frac{1}{2}}} \ \|\mathbf{q}\| \\
&\leq c_2^2 K_1^4 L L_1^2 \delta_1^2 \frac{2}{\nu\Lambda} + \frac{\nu}{8} \|\mathbf{q}\|^2 ,
\end{aligned}$$

and for the fourth, by using (12)

$$\begin{aligned}
|(\mathbf{B}(\mathbf{q},\mathbf{p}),\mathbf{q})| &\leq c_4 |\mathbf{q}|^{\frac{1}{2}} \|\mathbf{q}\|^{\frac{1}{2}} \|\mathbf{p}\|^{\frac{1}{2}} |\Delta\mathbf{p}|^{\frac{1}{2}} |\mathbf{q}| \leq \\
&\leq c_4 K_0^{1/2} L^{\frac{1}{4}} \delta^{\frac{1}{2}} K_1^{1/2} L^{\frac{1}{4}} \delta^{\frac{1}{4}} \rho_1^{\frac{1}{2}} \rho_2^{\frac{1}{2}} \frac{1}{\Lambda^{\frac{1}{2}}} \ \|\mathbf{q}\| \\
&\leq \frac{2}{\nu\Lambda} c_4^2 K_0 K_1 \rho_1 \rho_2 L \delta^{\frac{3}{2}} + \frac{\nu}{8} \|\mathbf{q}\|^2 .
\end{aligned}$$

At last

$$|(\mathbf{Qf},\mathbf{q})| \leq |\mathbf{Qf}| \, |\mathbf{q}| \leq \frac{2 |\mathbf{Qf}|^2}{\nu\Lambda} + \frac{\nu}{8} \|\mathbf{q}\|^2 .$$

The above inequalities and (23) lead us to

$$\frac{1}{2} \frac{d}{dt} |\mathbf{q}|^2 + 3\frac{\nu\Lambda}{8} |\mathbf{q}|^2 \leq C_0^2 \delta,$$

with

$$\begin{aligned}
C_0^2 &= \frac{2}{\nu\lambda} \Big[ c_1^2 \rho_0 \rho_2 K_1^2 L_1 \delta_1 + c_1^2 K_1 \rho_1^2 \rho_2 L_1^{1/2} \delta_1 + c_2^2 K_1^4 L L_1^2 \delta_1^2 + \\
&\quad + c_4^2 K_0 K_1 \rho_1 \rho_2 L \delta^{\frac{3}{2}} + |\mathbf{Qf}|^2 \Big] .
\end{aligned}$$

It follows, with the usual Gronwall Lemma,

$$|\mathbf{q}(t)|^2 \leq |\mathbf{q}(0)|^2 e^{-\frac{3}{4}\nu\Lambda t} + \frac{8C_0^2}{3\nu\lambda} \delta^2,$$

hence, for $t_4(R) \geq t_3(R)$, taken as to have $|\mathbf{q}(0)|^2 e^{-\frac{3}{4}\nu\Lambda t} \leq \frac{8C_0^2}{3\nu\lambda}\delta^2$ for $t \geq t_4(R)$, we obtain (18), with $\widetilde{C}_0 = \frac{4C_0}{\sqrt{3\nu\lambda}}$.

The functions of $n$: $L_1\delta_1 = L(n)\delta(n) = \left(1+\ln 2n^2\right)/(n+1)^2$, $L_1^{1/2}\delta_1 = \sqrt{L(n)}\delta(n) = \sqrt{1+\ln 2n^2}/(n+1)^2$, $LL_1^2\delta_1^2 = L(2n)L(n)^2\delta(n)^2 = \left(1+\ln 8n^2\right)\left(1+\ln 2n^2\right)^2/(n+1)^4$ and $L\delta^{\frac{3}{2}} = \left(1+\ln 8n^2\right)/(2n+1)^3$, that appear in the structure of $C_0^2$, have at $n=2$ values less than 1 and are decreasing when $n$ increases (for $n \geq 2$). Then, for $n \geq 2$

$$C_0^2 \leq \frac{2}{\nu\lambda} \left( c_1^2 \rho_0 \rho_2 K_1^2 + c_1^2 K_1 \rho_1^2 \rho_2 + c_2^2 K_1^4 + c_4^2 K_0 K_1 \rho_1 \rho_2 + |\mathbf{Qf}|^2 \right)$$

and the right hand side depends only on $\nu,\ \lambda,\ |\mathbf{Qf}|$. More than that, since all the functions defined above tend to zero when $n\to\infty$, we can choose $n$ large enough so that $\frac{2|\mathbf{Qf}|^2}{\nu\lambda}$ becomes the dominant term in $C_0^2$.

For that $n$, $\widetilde{C}_0$ will be of the order of $\frac{|\mathbf{Qf}|}{\nu\lambda}$. However, the structure of $K_0,\ K_1,\ \rho_0,\ \rho_1,\ \rho_2$ show that if $\nu$ is very small, then $n$ with the above property must be very large.

Now, we aim to estimate $\|\mathbf{q}\|$. By multiplying equation (14) by $\Delta\mathbf{q}$, and by using (22), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d\,\|\mathbf{q}\|^2}{dt}+\nu\,|\Delta\mathbf{q}|^2 \quad &\leq \quad \left|\left(\mathbf{B}(\mathbf{p}_p,\mathbf{p}_q+\mathbf{q}),\Delta\mathbf{q}\right)\right|+\left|\left(\mathbf{B}(\mathbf{p}_q+\mathbf{q},\mathbf{p}_p),\Delta\mathbf{q}\right)\right|+\\ &\quad +\left|\left(\mathbf{B}(\mathbf{p}_q+\mathbf{q},\mathbf{p}_q+\mathbf{q}),\Delta\mathbf{q}\right)\right|+\\ &\quad +\left|(\mathbf{Qf},\boldsymbol{\Delta}\mathbf{q})\right|.\end{aligned}$$

For the first term in the right hand side we have (7)

$$\begin{aligned}\left|\left(\mathbf{B}(\mathbf{p}_p,\mathbf{p}_q+\mathbf{q}),\Delta\mathbf{q}\right)\right| &\leq c_1\,|\mathbf{p}_p|^{\frac{1}{2}}\left|\boldsymbol{\Delta}\mathbf{p}_p\right|^{\frac{1}{2}}\|\mathbf{p}_q+\mathbf{q}\|\,|\Delta\mathbf{q}|\\ &\leq c_1\rho_0^{1/2}\rho_2^{1/2}L_1^{\frac{1}{2}}K_1\delta_1^{\frac{1}{2}}\,|\Delta\mathbf{q}|\\ &\leq c_1^2\rho_0\rho_2K_1^2L_1\delta_1\frac{2}{\nu}+\frac{\nu}{8}\,|\Delta\mathbf{q}|^2\,,\end{aligned}$$

for the second, with (12),

$$\begin{aligned}\left|\left(\mathbf{B}(\mathbf{p}_q+\mathbf{q},\mathbf{p}_p),\Delta\mathbf{q}\right)\right| &\leq c_4\,|\mathbf{p}_q+\mathbf{q}|^{\frac{1}{2}}\,\|\mathbf{p}_q+\mathbf{q}\|^{\frac{1}{2}}\,\|\mathbf{p}_p\|^{\frac{1}{2}}\,|\Delta\mathbf{p}_p|^{\frac{1}{2}}\,|\Delta\mathbf{q}|\\ &\leq c_4\widetilde{C}_0^{\frac{1}{2}}\delta_1^{\frac{1}{2}}L_1^{\frac{1}{4}}K_1^{\frac{1}{2}}\delta_1^{\frac{1}{4}}\rho_1^{\frac{1}{2}}\rho_2^{\frac{1}{2}}\,|\Delta\mathbf{q}|\\ &\leq c_4^2\widetilde{C}_0\rho_1\rho_2K_1L_1^{\frac{1}{2}}\delta_1^{\frac{3}{2}}\frac{2}{\nu}+\frac{\nu}{8}\,|\Delta\mathbf{q}|^2\end{aligned}$$

for the third, also with (12),

$$\begin{aligned}\left|\left(\mathbf{B}(\mathbf{p}_q+\mathbf{q},\mathbf{p}_q+\mathbf{q}),\Delta\mathbf{q}\right)\right| &\leq c_4\,|\mathbf{p}_q+\mathbf{q}|^{\frac{1}{2}}\,\|\mathbf{p}_q+\mathbf{q}\|\,|\Delta\left(\mathbf{p}_q+\mathbf{q}\right)|^{\frac{1}{2}}\,|\Delta\mathbf{q}|\\ &\leq c_4\widetilde{C}_0^{\frac{1}{2}}\delta_1^{\frac{1}{2}}L_1^{\frac{1}{2}}K_1\delta_1^{\frac{1}{2}}L_1^{\frac{1}{4}}K_2^{\frac{1}{2}}\,|\Delta\mathbf{q}|\\ &\leq c_4^2\widetilde{C}_0K_1^2K_2L_1^{\frac{3}{2}}\delta_1^2\frac{2}{\nu}+\frac{\nu}{8}\,|\Delta\mathbf{q}|^2\,,\end{aligned}$$

and for the fourth we have

$$|(\mathbf{Qf},\boldsymbol{\Delta}\mathbf{q})|\leq\frac{2\,|\mathbf{Qf}|^2}{\nu}+\frac{\nu}{8}\,|\Delta\mathbf{q}|^2\,.$$

We denote

$$\begin{aligned}\frac{1}{2}C_1^2 \quad &= \quad \frac{2c_1^2}{\nu}\rho_0\rho_2K_1^2L_1\delta_1+\frac{2c_4^2}{\nu}\widetilde{C}_0\rho_1\rho_2K_1L_1^{\frac{1}{2}}\delta_1^{\frac{3}{2}}+\\ &\quad +\frac{2c_4^2}{\nu}\widetilde{C}_0K_1^2K_2L_1^{\frac{3}{2}}\delta_1^2+\frac{2\,|\mathbf{Qf}|^2}{\nu}.\end{aligned}$$

Then the differential inequality for $\|\mathbf{q}\|$ becomes

$$\frac{d\|\mathbf{q}\|^2}{dt} + \nu\Lambda\|\mathbf{q}\|^2 \le C_1^2,$$

that yields

$$\|\mathbf{q}(t)\|^2 \le \|\mathbf{q}(0)\|^2 e^{-\nu\Lambda t} + \frac{1}{\nu\lambda} C_1^2 \delta.$$

Let $t_4(R)$ such that for $t \ge t_4(R) \ge t_3(R)$ the inequality $\|\mathbf{q}(0)\|^2 e^{-\nu\Lambda t} \le \frac{1}{\nu\lambda} C_1^2 \delta$ holds. For $t \ge t_4(R)$ (19) holds with $\widetilde{C}_1 = \sqrt{\frac{2}{\nu\lambda}} C_1$.

We remark that $L_1\delta_1 = L(n)\delta(n)$, $L_1^{\frac{1}{2}}\delta_1^{\frac{3}{2}} = L(n)^{\frac{1}{2}}\delta(n)^{\frac{3}{2}}$, $L_1^{\frac{3}{2}}\delta_1^2 = L(n)^{\frac{3}{2}}\delta(n)^2$ have values less than 1 for $n = 2$, decrease when $n$ increases for $n \ge 2$ and tend to zero when $n \to \infty$. Hence, $C_1$ may be replaced with a coefficient that depends only on $\nu$, $\lambda$, $|\mathbf{Qf}|$ and not on $n$.

Moreover, for $n$ large enough, each of the first four terms of $C_1$ becomes smaller than $\frac{|\mathbf{Qf}|}{\nu\sqrt{\lambda}}$.

As for the solution $\mathbf{u}$ in [13], it can be proved that $\mathbf{q}(t)$ is analytic in time and is the restriction to the real axis of an analytic function of complex variable defined on a neighborhood of the real axis, and by using the Cauchy formula, we obtain (20).

Finally, from (14) we have

$$\Delta\mathbf{q} = \frac{1}{\nu}\left[\frac{d\mathbf{q}}{dt} + \mathbf{QB}(\mathbf{p}+\mathbf{q}) - \mathbf{Qf}\right]$$

and with the above estimates we obtain (21).□

# 5 The new modified Galerkin method

Let us fix a $T > t_4(R)$. The interval $[0, T]$ is the interval on which we seek the approximate solution. Obviously, all the above inequalities are valid for $t \in [t_4(R), T]$.

In this section we just present the method, while in the following section we estimate the error of the method.

## 5.1 The first level

The first level of our method is related to the post-processed Galerkin method of [4].

Let $\mathbf{p}_0(t, \mathbf{x})$ be the solution of the equation (the Galerkin approximation of (13)):

$$\begin{aligned} \mathbf{p}_0' - \nu\boldsymbol{\Delta}\mathbf{p}_0 + \mathbf{PB}(\mathbf{p}_0) &= \mathbf{Pf}, \\ \mathbf{p}_0(0) &= \mathbf{Pu}_0, \end{aligned} \tag{24}$$

and

$$\mathbf{q}_0(t)= \mathbf{\Phi}_0\left(\mathbf{p}_0(t)\right),$$

where $\mathbf{\Phi}_0 : \mathbf{P}\mathcal{H} \to \mathbf{Q}\mathcal{H}$ is the function whose graph is the a.i.m. $\mathcal{M}_0$ defined in [3], that is

$$\mathbf{\Phi}_0\left(\mathbf{p}\right) = \left(\nu\mathbf{A}\right)^{-1}\left[\mathbf{Qf} - \mathbf{QB}\left(\mathbf{p}\right)\right]. \tag{25}$$

We define the corresponding approximate solution for the Navier-Stokes problem (5)-(6) as

$$\mathbf{u}_0\left(t\right) = \mathbf{p}_0\left(t\right) + \mathbf{q}_0\left(t\right). \tag{26}$$

Unlike the method of [4], we compute $\mathbf{q}_0$ at every moment of time, and not only at the end of the time interval, $T$ (in the course of the numerical implementation of this method, $\mathbf{q}_0$ will be computed at every point of the grid on $[0, T]$, constructed for the integration of (24)) .

**Remark.** In [15] the function $\mathbf{q}_{0,m}\left(t\right) = \mathbf{\Phi}_0\left(\mathbf{p}\left(t\right)\right)$ is defined (with $\mathbf{p}\left(t\right)$ the $\mathbf{P}$ projection of the exact solution), and then the function $\mathbf{u}_{0,m}\left(t\right) = \mathbf{p}\left(t\right) + \mathbf{q}_{0,m}\left(t\right)$, is constructed, it's positive trajectory being named "an induced trajectory". Since $\mathbf{p}_0\left(t\right)$ is an approximation of $\mathbf{p}\left(t\right)$, as is proved in Section 7, it follows that $\left\{\mathbf{u}_0\left(t\right);\ t \geq 0\right\}$, is an approximation of this first induced trajectory of [15].

## 5.2 The second level

The next level is different from both the nonlinear Galerkin methods and the post-processed Galerkin method already defined in literature. At this level we make use of $\mathbf{q}_0$ calculated in the previous step and define $\mathbf{p}_1$ as the solution of the equation:

$$\begin{aligned} \mathbf{p}_1' - \nu\mathbf{\Delta}\mathbf{p}_1 + \mathbf{PB}\left(\mathbf{p}_1 + \mathbf{q}_0\right) &= \mathbf{Pf}, \\ \mathbf{p}_1(0) &= \mathbf{Pu}_0. \end{aligned} \tag{27}$$

We expect this correction of $\mathbf{p}_0$ to be closer to $\mathbf{p}$ than $\mathbf{p}_0$ itself. Then we set

$$\begin{aligned} \mathbf{q}_1\left(t\right) = {} & \left(\nu\mathbf{A}\right)^{-1}\left[\mathbf{Qf} - \mathbf{QB}\left(\mathbf{p}_1\left(t\right)\right) - \mathbf{QB}\left(\mathbf{p}_1\left(t\right), \mathbf{q}_0\left(t\right)\right) - \right. \\ & \left. - \mathbf{QB}\left(\mathbf{q}_0\left(t\right), \mathbf{p}_1\left(t\right)\right)\right]. \end{aligned} \tag{28}$$

We define the approximate solution for (5)-(6) at this level by

$$\mathbf{u}_1(t) = \mathbf{p}_1(t) + \mathbf{q}_1(t). \tag{29}$$

**Remarks.**

**1.** In the non-linear Galerkin method [2], for the approximation $\mathbf{p}_1$ of $\mathbf{p}$, an equation, similar to (27), but with $\mathbf{PB}\left(\mathbf{p}_1 + \mathbf{\Phi}_0\left(\mathbf{p}_1\right)\right)$ instead of $\mathbf{PB}\left(\mathbf{p}_1 + \mathbf{q}_0\right)$ is considered.

**2.** In what concerns $\mathbf{q}_1$, the right hand side of (28) (let us denote it by $\widetilde{\Phi}_1(\mathbf{p}_1\left(t\right), \mathbf{q}_0\left(t\right))$ ) was inspired from the function $\mathbf{q}_{1,m}\left(t\right)$ of [15].

This one is defined as

$$\begin{aligned}\mathbf{q}_{1,m}(\mathbf{t}) &= (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}-\mathbf{QB}(\mathbf{p}(t))-\mathbf{QB}\left(\mathbf{p}(t),\mathbf{q}_{0,m}(t)\right)-\right.\\ &\quad \left.-\mathbf{QB}(\mathbf{q}_{0,m}(t),\mathbf{p}(t))\right]\end{aligned}$$

and with it's help the function $\mathbf{u}_{1,m}(\mathbf{t}) = \mathbf{p}(t)+\mathbf{q}_{1,m}(t)$, is defined, that generates a new induced trajectory. The construction of the second a.i.m. in [15], $\mathcal{M}_1$, is based upon the definition of function $\mathbf{q}_{1,m}(\mathbf{t})$. $\mathcal{M}_1$ is the graph of a function $\mathbf{\Phi}_1 : \mathbf{P}\mathcal{H} \to \mathbf{Q}\mathcal{H}$, given by

$$\begin{aligned}\mathbf{\Phi}_1(\mathbf{X}) &= (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}-\mathbf{QB}(\mathbf{X})-\mathbf{QB}(\mathbf{X},\mathbf{\Phi}_0(\mathbf{X}))-\right.\\ &\quad \left.-\mathbf{QB}(\mathbf{\Phi}_0(\mathbf{X}),\mathbf{X})\right].\end{aligned}$$

So, our function $\mathbf{u}_1(t)$ is related to an induced trajectory and, since this one is related to $\mathcal{M}_1$, it is also related to this a.i.m.

**3.** In the course of the numerical implementation of the method, $\mathbf{q}_1$ will be computed in the points of the grid on $[0,T]$, since it's values in these points will be used at the next level.

## 5.3 Inductive definition of the high-order approximations

Let us consider a $k\in\mathbb{N}$. We assume that for every $0\leq j\leq k+1$, we already constructed $\mathbf{p}_j$ and $\mathbf{q}_j$. Now, we define $\mathbf{p}_{k+2}$ as the solution of the problem

$$\begin{aligned}\mathbf{p}'_{k+2}-\nu\mathbf{\Delta}\mathbf{p}_{k+2}+\mathbf{PB}\left(\mathbf{p}_{k+2}+\mathbf{q}_{k+1}\right) &= \mathbf{Pf}, \\ \mathbf{p}_{k+2}(0) &= \mathbf{Pu}_0,\end{aligned}\tag{30}$$

with $\mathbf{q}_{k+1}$ defined at the preceding step, and then set $\mathbf{q}_{k+2}$ as

$$\begin{aligned}\mathbf{q}_{k+2} &= (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}-\mathbf{QB}(\mathbf{p}_{k+2})-\mathbf{QB}(\mathbf{p}_{k+2},\mathbf{q}_{k+1})-\right.\\ &\quad \left.-\mathbf{QB}(\mathbf{q}_{k+1},\mathbf{p}_{k+2})-\mathbf{QB}(\mathbf{q}_k,\mathbf{q}_k)-\mathbf{q}'_k\right].\end{aligned}\tag{31}$$

Naturally, the corresponding approximate solution of (5)-(6) is defined by

$$\mathbf{u}_{k+2}(t)=\mathbf{p}_{k+2}(t)+\mathbf{q}_{k+2}(t).$$

**Remarks.**

**1.** The right hand side of (31), that we denote by $\widetilde{\Phi}_{k+2}(\mathbf{p}_{k+2}(t),\mathbf{q}_{k+1}(t),\mathbf{q}_k(t))$, is inspired from inductive the definition of the function $\mathbf{q}_{k+2,m}(t)$ of [15], that is

$$\begin{aligned}\mathbf{q}_{k+2,m}(t) &= (\nu\mathbf{A})^{-1}\left[\mathbf{Qf}-\mathbf{QB}(\mathbf{p}(t))-\right.\\ &\quad -\mathbf{QB}(\mathbf{p}(t),\mathbf{q}_{k+1,m}(t))-\mathbf{QB}(\mathbf{q}_{k+1,m}(t),\mathbf{p}(t))-\\ &\quad \left.-\mathbf{QB}(\mathbf{q}_{k,m}(t))-\mathbf{q}'_{k,m}(t)\right].\end{aligned}$$

Our functions $\mathbf{u}_{k+2},\ k \geq 0$ are, in fact, approximations of the functions $\mathbf{u}_{k+2,m} = \mathbf{p} + \mathbf{q}_{k+2,m}$ that generate the induced trajectories in [15]. Our construction bypasses the construction of a.i.m.s. and is based directly upon that of the induced trajectories. We can call the sets $\{\mathbf{u}_{k+2}(t);\ t \geq 0\}$ - approximate induced trajectories.

**2.** The construction of the high accuracy a.i.m., $\mathcal{M}_{k+2}$, is based upon the definition of the function $\mathbf{q}_{k+2,m}$ of [15]. $\mathcal{M}_{k+2}$ is the graph of $\mathbf{\Phi}_{k+2} : \mathbf{P}\mathcal{H} \rightarrow \mathbf{Q}\mathcal{H}$,

$$\begin{aligned}\mathbf{\Phi}_{k+2}\left(\mathbf{X}\right) \quad = \quad & \left(\nu\mathbf{A}\right)^{-1}\left[\mathbf{Qf} - \mathbf{QB}\left(\mathbf{X}\right) - \right. \\ & -\mathbf{QB}\left(\mathbf{X}, \mathbf{\Phi}_{k+1}\left(\mathbf{X}\right)\right) - \mathbf{QB}\left(\mathbf{\Phi}_{k+1}\left(\mathbf{X}\right), \mathbf{X}\right) - \\ & \left. -\mathbf{QB}(\mathbf{\Phi}_{k}\left(\mathbf{X}\right)) - \mathbf{D\Phi}_{k}\left(\mathbf{X}\right)\Gamma_{k}\left(\mathbf{X}\right)\right]\end{aligned} \tag{32}$$

where $\mathbf{D\Phi}_k\left(\mathbf{X}\right)$ is the differential of $\mathbf{\Phi}_k\left(\mathbf{X}\right)$, and

$$\Gamma_k\left(\mathbf{X}\right) = \nu\mathbf{\Delta X} - \mathbf{PB}\left(\mathbf{X} + \mathbf{\Phi}_k\left(\mathbf{X}\right)\right) + \mathbf{Pf}.$$

These a.i.m.s or some variant of these are used in the nonlinear Galerkin methods, and in the postprocessed high-order nonlinear Galerkin methods.

**3.** If $k+2$ is the last level we construct, than we may compute $\mathbf{q}_{k+2}$ only at the moment of interest ($T$ for example) as in the postprocessed method of [4].

# 6 Estimates of the error of the approximate solutions

In the proof of the main result of this section, we need the following result that is a direct consequence of Lemma 1 from [4]. We denote by $\widehat{\mathbf{v}}^i_{j,l}$ the coordinate of the function $\mathbf{v}$ with respect to the eigenfunction $\mathbf{w}^i_{j,l}$

**Lemma** *Let* $\mathbf{G}(s) = \sum\limits_{j,l}\left(\sum\limits_{i=1}^{4}\widehat{G}^i_{j,l}\left(s\right)\mathbf{w}^i_{j,l}\right)$ *and suppose that*

$$\left|\widehat{\mathbf{G}}^i_{j,l}\left(s\right)\right| \leq c^i_{j,l}, \quad \mathit{for} \quad 0 \leq j, l \leq m,\ 1 \leq i \leq 4.$$

*Then*

$$\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\mathbf{PG}(s)ds\right| \leq \frac{1}{\nu}\left[\sum_{j,k\leq m}\sum_{i=1}^{4}\frac{\left(c^i_{j,l}\right)^2}{\lambda^2_{j,l}}\right]^{\frac{1}{2}}. \tag{33}$$

Now we can state and prove our main result.

**Theorem 2** *The functions* $\mathbf{u}_k\left(t\right),\ k \geq 0$, *defined in the previous section, represent approximate solutions of the problem* (1)-(3), *and their accuracy increases with* $k$. *More precisely, the inequality :*

$$|(\mathbf{u}-\mathbf{u}_k)(t)| \leq C\delta^{5/4+k/2}, \tag{34}$$

*holds for every* $k \geq 0$ *and for* $t \geq t_4(R)$.

**Proof** We will prove our assertion by induction.

**1.** We start with $k=0$. In [4] the following estimate is proved, for $\mathbf{f} \in \left[L^2_{per}(\Omega)\right]^2$ :

$$|(\mathbf{p}-\mathbf{p}_0)(t)| \leq \frac{C}{\Lambda^{5/4}} = C'\delta^{5/4}, \tag{35}$$

where $C'$ is a constant, large for $\nu$ small. Actually, as can be seen from [4] this $C'$ is of the order of the product $\widetilde{C}_0\widetilde{C}_1$, with $\widetilde{C}_0$, $\widetilde{C}_1$ the constants of our Theorem 1. Hence we can assume that $C'$ is of the form $K\frac{|\mathbf{Qf}|^2}{\nu^2\lambda^{3/2}}$ for $n$ great enough, with $K$ a number depending on $T$ but not on the data of the problem (see the proof of our Theorem 1).

Let us observe that $|\mathbf{p}_0(t)|$ is bounded for large times. Indeed,

$$\begin{aligned} |\mathbf{p}_0(t)| &= |\mathbf{p}(t)+\mathbf{p}_0(t)-\mathbf{p}(t)| \leq |\mathbf{p}(t)| + |\mathbf{p}_0(t)-\mathbf{p}(t)| \\ &\leq \rho_0 + C'\delta^{5/4} = \eta_0, \text{ for } t \text{ large enough.} \end{aligned}$$

The same observation is true for $\|\mathbf{p}_0(t)\|$ and for $|\Delta\mathbf{p}_0(t)|$

$$\begin{aligned} \|\mathbf{p}_0(t)\| &\leq \rho_1 + C'\delta^{3/4} = \eta_1, \\ |\Delta\mathbf{p}_0(t)| &\leq \rho_2 + C'\delta^{1/4} = \eta_2, \text{ for } t \text{ large enough.} \end{aligned}$$

In order to estimate the various norms of $(\mathbf{q}-\mathbf{q}_0)(t)$ we write

$$\mathbf{q} = (\nu\mathbf{A})^{-1}\left[\mathbf{Qf} - \mathbf{QB}(\mathbf{p}+\mathbf{q},\mathbf{p}+\mathbf{q}) + \frac{d\mathbf{q}}{dt}\right], \tag{36}$$

subtract from this relation the definition relation of $\mathbf{q}_0$, (25) and apply $\nu\mathbf{\Delta}$ to the obtained equality. In norm, we have

$$\begin{aligned} |\nu\mathbf{\Delta}(\mathbf{q}-\mathbf{q}_0)| &= \left|\mathbf{QB}(\mathbf{p}+\mathbf{q}) - \mathbf{QB}(\mathbf{p}_0) + \frac{d\mathbf{q}}{dt}\right| \\ &\leq |\mathbf{QB}(\mathbf{p}-\mathbf{p}_0,\mathbf{p})| + |\mathbf{QB}(\mathbf{p}_0,\mathbf{p}-\mathbf{p}_0)| + \\ &\quad + |\mathbf{QB}(\mathbf{p},\mathbf{q})| + |\mathbf{QB}(\mathbf{q},\mathbf{p})| + |\mathbf{QB}(\mathbf{q},\mathbf{q})| + \left|\frac{d\mathbf{q}}{dt}\right|. \end{aligned} \tag{37}$$

For the first term in the right side, with (8) we have:

$$\begin{aligned} |\mathbf{QB}(\mathbf{p}-\mathbf{p}_0,\mathbf{p})| &\leq c_2 L^{\frac{1}{2}} \|\mathbf{p}-\mathbf{p}_0\| \|\mathbf{p}\| \\ &\leq CL^{\frac{1}{2}}\rho_1\delta^{3/4} \leq C\rho_1\delta^{1/2}, \end{aligned}$$

where we used once more the inequality $L^{\frac{1}{2}}\delta^{1/4}\leq 1$. Here and in the sequel, $C$ denotes a generic constant (not depending on $m$ but depending on $\nu$, $\mathbf{f}$, $\lambda$).

The same estimate holds for the second term. With (7), the third term yields:

$$\begin{aligned}\left|\mathbf{QB}(\mathbf{p},\mathbf{q})\right| &\leq c_1\left|\mathbf{p}\right|^{\frac{1}{2}}\left|\mathbf{\Delta p}\right|^{\frac{1}{2}}\left\|\mathbf{q}\right\| \\ &\leq C\rho_0^{1/2}\rho_2^{1/2}\delta^{1/2},\end{aligned}$$

and the fourth

$$\begin{aligned}\left|\mathbf{QB}(\mathbf{q},\mathbf{p})\right| &\leq c_1\left|\mathbf{q}\right|^{\frac{1}{2}}\left|\mathbf{\Delta q}\right|^{\frac{1}{2}}\left\|\mathbf{p}\right\| \\ &\leq C\rho_1\delta^{1/2}.\end{aligned}$$

By using (20) and all the above inequalities in (37) we obtain

$$\left|\nu\Delta\left(\mathbf{q}\left(t\right)-\mathbf{q}_0\left(t\right)\right)\right|\leq C\delta^{1/2}, \tag{38}$$

for $t$ great enough. As consequences

$$\left\|\mathbf{q}\left(t\right)-\mathbf{q}_0\left(t\right)\right\|\leq C\delta,\ \ \left|\mathbf{q}\left(t\right)-\mathbf{q}_0\left(t\right)\right|\leq C\delta^{3/2}. \tag{39}$$

Inequality (35) and the second inequality above imply

$$\left|\mathbf{u}\left(t\right)-\mathbf{u}_0\left(t\right)\right|\leq C\delta^{5/4}. \tag{40}$$

We must remark that, as is proved for $\mathbf{u}$ in [13], we can prove that $\mathbf{p}_0$ is analytic in time, and more than that, it is the restriction of an analytic function of a complex variable to the real axis. This properties are transferred to $\mathbf{q}$ by its definition. Then, by using the Cauchy formula, it can be proved that

$$\left|\mathbf{q}'\left(t\right)-\mathbf{q}_0'\left(t\right)\right|\leq C\delta^{3/2}. \tag{41}$$

We also remark, for later use, that (25) and the dissipativity of $\mathbf{p}_0$ imply $\left|\Delta\mathbf{q}_0\right|\leq C$, $\left\|\mathbf{q}_0\right\|\leq C\delta^{\frac{1}{2}}$ and $\left|\mathbf{q}_0\right|\leq C\delta$ for $t\geq t_2\left(R\right)$.

**2.** We now estimate $\left|\mathbf{p}-\mathbf{p}_1\right|$ and $\left|\mathbf{q}-\mathbf{q}_1\right|$. We have, by subtracting (27) from (13)

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{p}-\mathbf{p}_1\right) &= \nu\mathbf{\Delta}\left(\mathbf{p}-\mathbf{p}_1\right)-\mathbf{PB}\left(\mathbf{p}+\mathbf{q}-\left(\mathbf{p_1}+\mathbf{q_0}\right),\mathbf{p}+\mathbf{q}\right)- \\ &\quad -\mathbf{PB}\left(\mathbf{p}_1+\mathbf{q}_0,\mathbf{p}+\mathbf{q}-\left(\mathbf{p_1}+\mathbf{q_0}\right)\right).\end{aligned}$$

From here, by using the semigroup of linear operators of infinitesimal generator $\nu\mathbf{A}$, we obtain

$$\begin{aligned}\frac{d}{dt}e^{\nu t\mathbf{A}}\left(\mathbf{p}-\mathbf{p}_1\right)(t) &= e^{\nu t\mathbf{A}}\left\{-\mathbf{PB}\left(\mathbf{p}-\mathbf{p}_1,\mathbf{u}\right)-\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{u}\right)-\right. \\ &\quad \left.-\mathbf{PB}\left(\mathbf{p}_1+\mathbf{q}_0,\mathbf{q}-\mathbf{q}_0\right)-\mathbf{PB}\left(\mathbf{p}_1+\mathbf{q}_0,\mathbf{p}-\mathbf{p}_1\right)\right\},\end{aligned}$$

and, by integrating

$$\begin{aligned}
\left(\mathbf{p}-\mathbf{p}_1\right)(t) &= e^{-\nu t\mathbf{A}}\left(\mathbf{p}-\mathbf{p}_1\right)(0)- \\
&-\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{p}-\mathbf{p}_1,\mathbf{u}\right)+\mathbf{PB}\left(\mathbf{p}_1+\mathbf{q}_0,\mathbf{p}-\mathbf{p}_1\right)\right\}ds- \\
&-\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{p}_1+\mathbf{q}_0,\mathbf{q}-\mathbf{q}_0\right)+\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{u}\right)\right\}ds.
\end{aligned}$$

Following [4] we use the inequalities [1]

$$\left|\mathbf{A}^{-\delta}\mathbf{B}\left(\mathbf{u},\mathbf{v}\right)\right| \le \begin{cases} C\left|\mathbf{A}^{1-\delta}\mathbf{u}\right|\left|\mathbf{v}\right| \le C\left|\mathbf{A}^{1/2}\mathbf{u}\right|\left|\mathbf{v}\right|, \\ C\left|\mathbf{u}\right|\left|\mathbf{A}^{1-\delta}\mathbf{v}\right| \le C\left|\mathbf{u}\right|\left|\mathbf{A}^{1/2}\mathbf{v}\right|, \end{cases}$$

valid for $\delta\in(1/2,1)$ and [7]

$$\left|\mathbf{A}^{\delta}e^{-\nu t\mathbf{A}}\right| \le Ct^{-\delta}e^{-\frac{\nu\lambda}{2}t},$$

and obtain

$$\begin{aligned}
\left|\left(\mathbf{p}-\mathbf{p}_1\right)(t)\right| \le \left|e^{-\nu t\mathbf{A}}\left(\mathbf{p}-\mathbf{p}_1\right)(0)\right|+\textstyle\int_0^t C\left(t-s\right)^{-\delta}e^{-\frac{\nu\lambda}{2}(t-s)}\left|\left(\mathbf{p}-\mathbf{p}_1\right)(s)\right|ds+ \\
+\left|\textstyle\int_0^t e^{-\nu(t-s)\mathbf{A}}\left[\mathbf{PB}\left(\mathbf{p}_1+\mathbf{q}_0,\mathbf{q}-\mathbf{q}_0\right)+\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{p}+\mathbf{q}\right)\right](s)\,ds\right|.
\end{aligned}$$

A form of Gronwall inequality ([7], Lemma 7.1.1) implies
$\left|\left(\mathbf{p}-\mathbf{p}_1\right)(t)\right| \le$
$\le C\max\limits_{0\le t\le T}\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{p}_1+\mathbf{q}_0,\mathbf{q}-\mathbf{q}_0\right)+\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{p}+\mathbf{q}\right)\right\}(s)\,ds\right|.$
We must remark that the constant $C$ above is of the order of $e^T$.

By using the method of [4], we find the estimates for the coordinates of the several terms in the accolade:

$$\left|\widehat{\mathbf{B}\left(\mathbf{q}_0,\mathbf{q}-\mathbf{q}_0\right)}_{j,k}\right| \le \left|\mathbf{q}_0\right|\left|\mathbf{A}^{\frac12}\left(\mathbf{q}-\mathbf{q}_0\right)\right| \le C\delta\delta = C\delta^2, \tag{42}$$

$$\left|\widehat{\mathbf{B}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{q}\right)}_{j,k}\right| \le \left|\mathbf{q}-\mathbf{q}_0\right|\left|\mathbf{A}^{\frac12}\mathbf{q}\right| \le C\delta^{3/2}\delta^{1/2} = C\delta^2, \tag{43}$$

$$\begin{aligned}
\left|\widehat{\mathbf{B}\left(\mathbf{p}_1,\mathbf{q}-\mathbf{q}_0\right)}_{j,k}\right| &\le \left|\mathbf{A}^{\frac12}\left(\mathbf{q}-\mathbf{q}_0\right)\right|\left(\left|\left(\mathbf{I}-\mathbf{P}_{m-j}\right)\mathbf{p}\right|+\left|\left(\mathbf{I}-\mathbf{P}_{m-k}\right)\mathbf{p}\right|\right) \\
&\le C\delta\left(\frac{1}{\lambda_{m-j+1}}+\frac{1}{\lambda_{m-k+1}}\right),
\end{aligned} \tag{44}$$

$$\left|\widehat{\mathbf{B}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{p}\right)}_{j,k}\right| \le CK\delta^{3/2}\left(\frac{1}{\lambda^{\frac12}_{m-j+1}}+\frac{1}{\lambda^{\frac12}_{m-k+1}}\right),$$

where $\mathbf{P}_{m-j}$ represents the projection operator on the space spanned by the eigenfunctions corresponding to the eigenvalues in $\Gamma_{m-j}$ and $\lambda_j=\lambda_{j,0}$.

By using the inequalities (33), (42), (43) and

$$\sum_{j,k\le m}\lambda_{j,k}^{-2}\le\widetilde{C},$$

it follows that

$$\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{q}_0,\mathbf{q}-\mathbf{q}_0\right)+\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{q}\right)\right\}ds\right|\leq\widetilde{C}K\delta^2.$$

In order to estimate the term $\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\mathbf{PB}\left(\mathbf{p}_1,\mathbf{q}-\mathbf{q}_0\right)ds\right|$ we use (44) and the inequality

$$\sum_{j,k\leq m}\frac{1}{\lambda_{j,k}^2\lambda_{m-j+1}^2}\leq\frac{C}{(m+1)^3}=C\delta^{3/2},$$

proved in [4]. It follows

$$\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\mathbf{PB}\left(\mathbf{p}_1,\mathbf{q}-\mathbf{q}_0\right)ds\right|\leq C\delta^{1+\frac{3}{4}}.$$

The same estimate can be proved for $\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_0,\mathbf{p}\right)\right\}ds\right|$, hence finally we have

$$\left|\mathbf{p}-\mathbf{p}_1\right|\leq C\delta^{7/4}. \tag{45}$$

We easily see that $|\mathbf{p}_1|\leq\eta_0,\ \|\mathbf{p}_1\|\leq\eta_1,\ |\Delta\mathbf{p}_1|\leq\eta_2$.

Now, in order to estimate the various norms of $\mathbf{q}-\mathbf{q}_1$, we subtract (28) from (36), we apply the operator $\nu\Delta$, and take the norm in $\mathcal{H}$ of the resulted equality. After grouping the terms in a convenient way, we get

$$\begin{aligned}\left|\nu\boldsymbol{\Delta}\left(\mathbf{q}-\mathbf{q}_1\right)\right| \leq\ & \left|\mathbf{QB}\left(\mathbf{p}-\mathbf{p}_1,\mathbf{p}\right)\right|+\left|\mathbf{QB}\left(\mathbf{p}_1,\mathbf{p}-\mathbf{p}_1\right)\right|+\\ &+\left|\mathbf{QB}\left(\mathbf{p}-\mathbf{p}_1,\mathbf{q}\right)\right|+\left|\mathbf{QB}\left(\mathbf{q}_0,\mathbf{p}-\mathbf{p}_1\right)\right|\\ &+\left|\mathbf{QB}\left(\mathbf{p}_1,\mathbf{q}-\mathbf{q}_0\right)\right|+\left|\mathbf{QB}\left(\mathbf{q}-\mathbf{q_0},\mathbf{p}\right)\right|\\ &+\left|\mathbf{QB}\left(\mathbf{q},\mathbf{q}\right)\right|+\left|\frac{d\mathbf{q}}{dt}\right|.\end{aligned} \tag{46}$$

As we did for $|\mathbf{q}-\mathbf{q}_0|$, we estimate one by one the terms from the right side. For the first one, we use (8) and the inequality $L^{1/2}\delta^{1/4}\leq 1:$

$$\begin{aligned}\left|\mathbf{QB}\left(\mathbf{p}-\mathbf{p}_1,\mathbf{p},\right)\right| &\leq c_2L^{1/2}\left\|\mathbf{p}-\mathbf{p}_1\right\|\left\|\mathbf{p}\right\|\\ &\leq CL^{1/2}\delta^{5/4}\rho_1\leq C\delta.\end{aligned}$$

The same estimate is valid for the second term. The third term is smaller than the first and for the fourth the following holds

$$\begin{aligned}\left|\mathbf{QB}\left(\mathbf{q}_0,\mathbf{p}-\mathbf{p}_1\right)\right| &\leq c_1\left|\mathbf{q_0}\right|^{\frac{1}{2}}\left|\Delta\mathbf{q_0}\right|^{\frac{1}{2}}\left\|\mathbf{p}-\mathbf{p}_1\right\|\\ &\leq C\delta^{1/2}\delta^{5/4}=C\delta^{7/4}.\end{aligned}$$

For the two following terms we use (8) respectively (11):

$$\begin{aligned}\left|\mathbf{B}\left(\mathbf{p}_1,\mathbf{q}-\mathbf{q_0}\right)\right| &\leq c_1\left|\mathbf{p}_1\right|^{\frac{1}{2}}\left|\boldsymbol{\Delta}\mathbf{p}_1\right|^{\frac{1}{2}}\left\|\mathbf{q}-\mathbf{q_0}\right\| \\ &\leq C\eta_0^{1/2}\eta_2^{1/2}\delta,\end{aligned}$$

$$\begin{aligned}\left|\mathbf{B}\left(\mathbf{q}-\mathbf{q_0},\mathbf{p}\right)\right| &\leq c_4\left|\mathbf{q}-\mathbf{q_0}\right|^{\frac{1}{2}}\left\|\mathbf{q}-\mathbf{q_0}\right\|^{\frac{1}{2}}\left\|\mathbf{p}\right\|^{\frac{1}{2}}\left|\Delta\mathbf{p}\right|^{\frac{1}{2}} \\ &\leq C\delta^{3/4}\delta^{1/2}\rho_1^{1/2}\rho_2^{1/2}\leq C\delta^{5/4}.\end{aligned}$$

The fifth and sixth terms are smaller than the first, respectively the second term, while for the seventh we have, with (11)

$$\begin{aligned}\left|\mathbf{B}\left(\mathbf{q},\mathbf{q}\right)\right| &\leq c_4\left|\mathbf{q}\right|^{\frac{1}{2}}\left\|\mathbf{q}\right\|^{\frac{1}{2}}\left\|\mathbf{q}\right\|^{\frac{1}{2}}\left|\Delta\mathbf{q}\right|^{\frac{1}{2}} \\ &\leq C\delta.\end{aligned}$$

By using the above inequalities in (46) we obtain

$$\left|\nu\boldsymbol{\Delta}\left(\mathbf{q}-\mathbf{q}_1\right)\right|\leq C\delta.$$

From here

$$\left\|\mathbf{q}-\mathbf{q}_1\right\|\leq C\delta^{3/2},\qquad \left|\mathbf{q}-\mathbf{q}_1\right|\leq C\delta^2. \tag{47}$$

The arguments used to state the analyticity in time of $\mathbf{q_0}$ remain valid for $\mathbf{q_1}$ and the following relation follows

$$\left|\mathbf{q}'-\mathbf{q}_1'\right|\leq C\delta^2.$$

This will be used later. By using (45) and (47) we now obtain

$$\left|\mathbf{u}-\mathbf{u}_1\right|\leq CL\delta^{7/4}. \tag{48}$$

**3. The induction step.** We assume that, for every $0\leq j\leq k+1$ the inequalities

$$\begin{aligned}\left|\mathbf{p}-\mathbf{p}_j\right| &\leq C\delta^{5/4+j/2}, \\ \left|\mathbf{q}-\mathbf{q}_j\right| &\leq C'\delta^{3/2+j/2}, \\ \left|\mathbf{q}'-\mathbf{q}_j'\right| &\leq C''\delta^{3/2+j/2} \\ \left\|\mathbf{q}-\mathbf{q}_j\right\| &\leq C'''\delta^{1+j/2}.\end{aligned}$$

hold. We prove that the above inequalities hold also for $j=k+2$:

$$\begin{gathered}\left(\mathbf{p}-\mathbf{p}_{k+2}\right)(t)=e^{-\nu t\mathbf{A}}\left(\mathbf{p}-\mathbf{p}_{k+2}\right)(0)- \\ -\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{p}-\mathbf{p}_{k+2},\mathbf{u}\right)+\mathbf{PB}\left(\mathbf{p}_{k+2}+\mathbf{q}_{k+1},\mathbf{p}-\mathbf{p}_{k+2}\right)\right\}ds- \\ -\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{p}_{k+2}+\mathbf{q}_{k+1},\mathbf{q}-\mathbf{q}_{k+1}\right)+\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_{k+1},\mathbf{u}\right)\right\}ds.\end{gathered}$$

As we did for $|(\mathbf{p}-\mathbf{p}_1)(t)|$, we obtain
$\left|\left(\mathbf{p}-\mathbf{p}_{k+2}\right)(t)\right| \leq \left|e^{-\nu t\mathbf{A}}\left(\mathbf{p}-\mathbf{p}_{k+2}\right)(0)\right| + \int_0^t C\,(t-s)^{-\delta}\, e^{-\frac{\nu\lambda}{2}(t-s)} \left|\mathbf{p}-\mathbf{p}_{k+2}\right| ds +$
$+ \left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{p}_{k+2}+\mathbf{q}_{k+1}, \mathbf{q}-\mathbf{q}_{k+1}\right) + \mathbf{PB}\left(\mathbf{q}-\mathbf{q}_{k+1}, \mathbf{p}+\mathbf{q}\right)\right\} ds\right|.$
The already cited Gronwall-type Lemma of [7] implies
$\left|\left(\mathbf{p}-\mathbf{p}_{k+2}\right)(t)\right| \leq$
$\leq C \max\limits_{0\leq t\leq T} \left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{p}_{k+2}+\mathbf{q}_{k+1}, \mathbf{q}-\mathbf{q}_{k+1}\right) + \mathbf{PB}\left(\mathbf{q}-\mathbf{q}_{k+1}, \mathbf{p}+\mathbf{q}\right)\right\} ds\right|.$
We evaluate the coordinates of each term in the brackets following $e^{-\nu(t-s)\mathbf{A}}$:

$$\begin{aligned}
\left|B\widehat{\left(\mathbf{q}_{k+1}, \mathbf{q}-\mathbf{q}_{k+1}\right)}_{j,l}\right| &\leq |\mathbf{q}_{k+1}| \left|\mathbf{A}^{\frac{1}{2}}\left(\mathbf{q}-\mathbf{q}_{k+1}\right)\right| \leq C\delta\delta^{3/2+k/2} = C\delta^{3/2+(k+2)/2},\\
\left|B\widehat{\left(\mathbf{q}-\mathbf{q}_{k+1}, \mathbf{q}\right)}_{j,l}\right| &\leq |\mathbf{q}-\mathbf{q}_{k+1}| \left|\mathbf{A}^{\frac{1}{2}}\mathbf{q}\right| \leq C\delta^{3/2+(k+1)/2}\delta^{\frac{1}{2}} = C\delta^{3/2+(k+2)/2},\\
\left|B\widehat{\left(\mathbf{p}_{n+2}, \mathbf{q}-\mathbf{q}_{n+1}\right)}_{j,l}\right| &\leq \left|\mathbf{A}^{\frac{1}{2}}\left(\mathbf{q}-\mathbf{q}_{n+1}\right)\right| \left(\left|\left(\mathbf{I}-\mathbf{P}_{m-j}\right)\mathbf{p}_{n+2}\right| + \left|\left(\mathbf{I}-\mathbf{P}_{m-l}\right)\mathbf{p}_{n+2}\right|\right)\\
&\leq C\delta^{3/2+n/2}\left(1/\lambda_{m-j+1} + 1/\lambda_{m-l+1}\right),\\
\left|B\widehat{\left(\mathbf{q}-\mathbf{q}_{n+1}, \mathbf{p}\right)}_{j,l}\right| &\leq C\delta^{3/2+(n+1)/2}\left(1/\lambda^{\frac{1}{2}}_{m-j+1} + 1/\lambda^{\frac{1}{2}}_{m-l+1}\right).
\end{aligned}$$

The same arguments used for the terms involved in $|(\mathbf{p}-\mathbf{p}_1)(t)|$ lead to

$$\begin{aligned}
\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{q}_{k+1}, \mathbf{q}-\mathbf{q}_{k+1}\right) + \mathbf{PB}\left(\mathbf{q}-\mathbf{q}_{k+1}, \mathbf{q}\right)\right\} ds\right| &\leq C\delta^{3/2+(k+2)/2},\\
\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\mathbf{PB}\left(\mathbf{p}_{k+2}, \mathbf{q}-\mathbf{q}_{k+1}\right) ds\right| &\leq C\delta^{(3+k)/2+3/4}\\
&= C\delta^{(k+2)/2+5/4}.
\end{aligned}$$

Analogously we can show that

$$\left|\int_0^t e^{-\nu(t-s)\mathbf{A}}\left\{\mathbf{PB}\left(\mathbf{q}-\mathbf{q}_{k+1}, \mathbf{p}\right)\right\} ds\right| \leq C\delta^{(k+2)/2+5/4},$$

and by putting these results together, it follows

$$\left|\left(\mathbf{p}-\mathbf{p}_{k+2}\right)(t)\right| \leq C\delta^{(k+2)/2+5/4}, \tag{49}$$

that confirms our induction hypothesis in what concerns $\mathbf{p}_k$. It also follows that

$$|\mathbf{p}_{k+2}| \leq \eta_0, \quad \|\mathbf{p}_{k+2}\| \leq \eta_1, \quad |\Delta\mathbf{p}_{k+2}| \leq \eta_2.$$

Now for $\left|\nu\mathbf{\Delta}\left(\mathbf{q}-\mathbf{q}_{k+2}\right)(t)\right|$ we have

$$\begin{aligned}
\nu\mathbf{\Delta}\left(\mathbf{q}-\mathbf{q}_{k+2}\right) = {} & \mathbf{QB}(\mathbf{p}) - \mathbf{QB}(\mathbf{p}_{k+2}) + \mathbf{QB}(\mathbf{p},\mathbf{q}) - \mathbf{QB}(\mathbf{p}_{k+2},\mathbf{q}_{k+1}) +\\
& + \mathbf{QB}(\mathbf{q},\mathbf{p}) - \mathbf{QB}(\mathbf{q}_{k+1},\mathbf{p}_{k+2}) + \mathbf{QB}(\mathbf{q},\mathbf{q}) - \mathbf{QB}(\mathbf{q}_k,\mathbf{q}_k) +\\
& + \mathbf{q}' - \mathbf{q}'_k
\end{aligned}$$

and

$$\begin{aligned}
\left|\nu\boldsymbol{\Delta}\left(\mathbf{q}-\mathbf{q}_{k+2}\right)\right| \quad &\leq \quad \left|\mathbf{QB}(\mathbf{p}-\mathbf{p}_{k+2},\mathbf{p})+\mathbf{QB}(\mathbf{p}_{k+2},\mathbf{p}-\mathbf{p}_{k+2})\right|+\\
&\quad +\left|\mathbf{QB}(\mathbf{p}-\mathbf{p}_{k+2},\mathbf{q}_{k+1})+\mathbf{QB}(\mathbf{q_{k+1}},\mathbf{p}-\mathbf{p_{k+2}}))\right|+\\
&\quad +\left|\mathbf{QB}(\mathbf{q}-\mathbf{q}_{k+1},\mathbf{p})+\mathbf{QB}(\mathbf{p_{k+2}},\mathbf{q}-\mathbf{q_{k+1}})\right|+\\
&\quad +\left|\mathbf{QB}(\mathbf{q}-\mathbf{q}_{k},\mathbf{q})+\mathbf{QB}(\mathbf{q}_{k},\mathbf{q}-\mathbf{q}_{k})\right|+\left|\mathbf{q}'-\mathbf{q}'_{k}\right|.
\end{aligned}$$

We can see, by using the induction hypothesis, (49) and $L^{1/2}\delta^{1/4}\leq 1$, that, for the first two terms, we have

$$\begin{aligned}
\left.\begin{array}{r}\left|\mathbf{QB}(\mathbf{p}-\mathbf{p}_{k+2},\mathbf{p})\right|\\ \left|\mathbf{QB}(\mathbf{p}_{k+2},\mathbf{p}-\mathbf{p}_{k+2})\right|\end{array}\right\} \quad &\leq \quad c_2L^{1/2}\left\|\mathbf{p}-\mathbf{p}_{k+2}\right\|\eta_1\\
&\leq \quad CL^{1/2}\eta_1\delta^{5/4+(k+1)/2}\\
&\leq \quad C\eta_1\delta^{1/4+(k+2)/2}.
\end{aligned}$$

The following term is smaller than the first. The fourth term can be estimated as follows

$$\begin{aligned}
\left|\mathbf{QB}(\mathbf{q_{k+1}},\mathbf{p}-\mathbf{p_{k+2}})\right| \quad &\leq \quad c_1\left|\mathbf{q}_{k+1}\right|^{\frac{1}{2}}\left|\boldsymbol{\Delta}\mathbf{q}_{k+1}\right|^{\frac{1}{2}}\left\|\mathbf{p}-\mathbf{p}_{k+2}\right\|\\
&\leq \quad C\delta^{1/2}\delta^{5/4+(k+1)/2}=C\delta^{5/4+(k+2)/2}.
\end{aligned}$$

For the fifth term, we obtain

$$\begin{aligned}
\left|\mathbf{QB}(\mathbf{q}-\mathbf{q}_{k+1},\mathbf{p})\right| \quad &\leq \quad c_4\left|\mathbf{q}-\mathbf{q}_{k+1}\right|^{\frac{1}{2}}\left\|\mathbf{q}-\mathbf{q}_{k+1}\right\|^{\frac{1}{2}}\left\|\mathbf{p}\right\|^{\frac{1}{2}}\left|\Delta\mathbf{p}\right|^{\frac{1}{2}}\\
&\leq \quad C\rho_1^{1/2}\rho_2^{1/2}\delta^{3/4+(k+1)/4}\delta^{3/4+k/4}\\
&\leq \quad C\rho_1^{1/2}\rho_2^{1/2}\delta^{3/2+(2k+1)/4},
\end{aligned}$$

and for the sixth

$$\begin{aligned}
\left|\mathbf{QB}(\mathbf{p}_{k+2},\mathbf{q}-\mathbf{q}_{k+1})\right| \quad &\leq \quad c_1\left|\mathbf{p}_{k+2}\right|^{\frac{1}{2}}\left|\boldsymbol{\Delta}\mathbf{p}_{k+2}\right|^{\frac{1}{2}}\left\|\mathbf{q}-\mathbf{q}_{k+1}\right\|\\
&\leq \quad c_1\delta\eta_0^{1/2}\eta_2^{1/2}\delta^{3/2+k/2}\\
&\leq \quad c_1\eta_0^{1/2}\eta_2^{1/2}\delta^{3/2+(k+2)/2}.
\end{aligned}$$

Then, by using (12) we obtain

$$\begin{aligned}
\left|\mathbf{QB}(\mathbf{q}-\mathbf{q}_{k},\mathbf{q})\right| \quad &\leq \quad c_4\left|\mathbf{q}-\mathbf{q}_{k+1}\right|^{\frac{1}{2}}\left\|\mathbf{q}-\mathbf{q}_{k+1}\right\|^{\frac{1}{2}}\left\|\mathbf{q}\right\|^{\frac{1}{2}}\left|\Delta\mathbf{q}\right|^{\frac{1}{2}}\\
&\leq \quad C\delta^{3/4+(k+1)/4}\delta^{3/4+k/4}\delta^{1/4}=C\delta^{1+(k+2)/2},\\
\left|\mathbf{QB}(\mathbf{q}_{k},\mathbf{q}-\mathbf{q}_{k})\right| \quad &\leq \quad c_4\left|\mathbf{q}_{k}\right|^{\frac{1}{2}}\left\|\mathbf{q}_{k}\right\|^{\frac{1}{2}}\left\|\mathbf{q}-\mathbf{q}_{k}\right\|^{\frac{1}{2}}\left|\Delta\left(\mathbf{q}-\mathbf{q}_{k}\right)\right|^{\frac{1}{2}}\\
&\leq \quad C\delta^{1/2}\delta^{1/4}\delta^{3/4+k/4}\delta^{3/4+(k-1)/4}=C\delta^{1+(k+2)/2}.
\end{aligned}$$

By using also the induction hypothesis on $|\mathbf{q}'-\mathbf{q}'_k|$ and by comparing the magnitude orders of the various terms, we find, successively,

$$\begin{aligned} |\nu\Delta(\mathbf{q}-\mathbf{q}_{k+2})| &\leq C\delta^{1/2+(k+2)/2}, \\ \|\mathbf{q}-\mathbf{q}_{k+2}\| &\leq C\delta^{1+(k+2)/2}, \\ |\mathbf{q}-\mathbf{q}_{k+2}| &\leq C\delta^{3/2+(k+2)/2}, \end{aligned}$$

and these inequalities confirm our induction hypothesis.

From (49) and the above estimates it follows

$$|\mathbf{u}-\mathbf{u}_{k+2}| \leq C\delta^{5/4+(k+2)/2}.\square \tag{50}$$

# 7 Comments on the method

**1.** A major advantage of our method is that we can use very low dimensional projection spaces for the approximations of $\mathbf{p}$, since the accuracy of the approximate solution may be increased by using several iteration levels of the method.

For example, if we choose $m=6$, after having passed through five levels of the method we obtain an approximate solution $\mathbf{u}_4(t)=\mathbf{p}_4(t)+\mathbf{q}_4(t)$ that bears an error of the order of $10^{-5}$ since $\delta^{13/4}=\frac{1}{7^{13/2}}\simeq 0.0000032$. At each level $j,\ 0\leq j\leq 4$ we will have to solve a system of $4\times 36+4\times 6=168$ ODEs, on the interval $[0,T]$ in order to find the coordinates of $\mathbf{p}_j(t)$, and to compute the coordinates of $\mathbf{q}_j(t)$ by using algebraic relations.

Here, as in all nonlinear Galerkin methods, problems appear due to $\mathbf{f}$. If this function has a infinity of nonzero coefficients in its Fourier function, it will generate a infinite number of non-zero coordinates in $\mathbf{q}_j(t)$. A truncation criterion must be applied and it will depend on $\mathbf{f}$. Thus the number of coordinates of $\mathbf{q}_j(t)$ to be computed depends on $j$ and on the given function $\mathbf{f}$.

If we chose $m=10$, we need only four levels of the method for an error of the order of $10^{-5}$ (in this case $\delta^{11/4}=\frac{1}{11^{11/2}}\simeq 0.00000187$). But at each level a number of $4\times 100+4\times 10=440$ ODEs must be solved for the coordinates of $\mathbf{p}_j(t)$. Besides these, at each level $j$ the coordinates of $\mathbf{q}_j(t)$ must be computed by algebraic relations resulted from the definitions.

**2.** The program for the integration the systems of ODEs for $\mathbf{p}_j(t)$ should have the same structure for all $j$, only the coordinates of $\mathbf{q}_{j-1}(t)$ remaining to be replaced in the nonlinear term.

**3.** A comparison with the nonlinear Galerkin methods that use high-accurate a.i.m.s is necessary.

The nonlinear Galerkin method based on the use of high accuracy a.i.m.s [2], [11], applied to the Navier-Stokes problem and corresponding to our level $k+2,\ k\geq 0$, consists in solving the finite dimensional problem

$$\begin{aligned} \frac{d\widetilde{\mathbf{p}}}{dt}-\nu\boldsymbol{\Delta}\widetilde{\mathbf{p}}+\mathbf{PB}(\widetilde{\mathbf{p}}+\boldsymbol{\Phi}_{k+1}(\widetilde{\mathbf{p}})) &= \mathbf{Pf}, \\ \widetilde{\mathbf{p}}(0) &= \mathbf{Pu}(0), \end{aligned} \tag{51}$$

for the approximation $\widetilde{\mathbf{p}}$ of $\mathbf{p} = \mathbf{Pu}$. Here, as above, $\mathbf{\Phi}_{k+1} : \mathbf{P}\mathcal{H} \to \mathbf{Q}\mathcal{H}$, is the function defining an a.i.m. of high accuracy. The advantage of this method towards ours is that the system of equations for $\widetilde{\mathbf{p}}$ is integrated only once. But the problem with solving (51) is that the definition of $\mathbf{\Phi}_{k+1}$ requires those of all $\mathbf{\Phi}_j$ with $j < k+1$ and is very laborious (see [2]). Programming this must be very difficult. The structure of our method, with iterative levels, makes the computations easier to program, and each level represents a certain approximation of the solution, so we can enjoy partial results.

On another hand, all the computations for $\mathbf{q}_0(t)$, $\mathbf{q}_1(t), ..., \mathbf{q}_{k+1}(t)$ (together) in our method seem, at first glance, of the same order of complexity as those necessary for the evaluation of $\mathbf{\Phi}_{k+1}(\widetilde{\mathbf{p}}(t))$ in the course of the numerical integration of (51) in [2], [11]. However, in our method, a major simplification of the computations appears since $\mathbf{q}'_{k-2}$ (from the definition of $\mathbf{q}_k$) may be approximated by the numerical derivative $(\mathbf{q}_{k-2}(t) - \mathbf{q}_{k-2}(t-h))/h$ (since we have already computed $\mathbf{q}_{k-2}(t)$ at every time step). This must be compared with the definitions of $z'_{j,m}$ in [2] or $q_j^1$ in [11], that yield difficulties in the numerical integration programming. We must also remark here that the term $\mathbf{D\Phi}_{k-1}(\mathbf{X})\,\Gamma_{k-1}(\mathbf{X})$ in the definition of $\mathbf{\Phi}_{k+1}(\mathbf{X})$ (that is (32) with $k+1$ instead of $k+2$) is meant to approximate $\mathbf{q}'_{k-1,m}$ from the definition of $\mathbf{q}_{k+1,m}$. Hence, conceiving a method that uses directly the functions $\mathbf{q}_{k+1,m}$ instead of the a.i.m.s that are defined with the help of these functions is very natural.

**4.** The memory of the computer is better organized in our method, since at the beginning of the computations for the level $j$ we may erase from the memory the value of $\mathbf{p}_{j-1}(t)$ and keep only those of $\mathbf{q}_{j-1}(t)$, $\mathbf{q}_{j-2}(t)$.

**5.** In order to have not too many computations we may postprocess our solution $\mathbf{p}_{k+2}(t)$ (at the last level) only at the end of the time interval $[0, T]$, by adding $\mathbf{q}_{k+2}(T) = \widetilde{\Phi}_{k+2}(\mathbf{p}_{k+2}(T), \mathbf{q}_{k+1}(T), \mathbf{q}_k(T))$, as is done in [11] for (51).

**Acknowledgements**

Work supported by the Romanian Ministry of Education and Research via the grant CEEX-05-D11-25/5.11.2005

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